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STABILITY OF CIRCULAR CYLINDRICAL SHELLS OF VARIABLE THICKNESS FOR A BENDING STATE OF STRESS

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Kh. K. SEIFULLAEV
(Baku)

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The stability problem of circular cylindrical shells of variable thickness under axial compression is examined, taking account of the bending stress of the initial pre-critical state.

The initial bending equilibrium states of shells of variable thickness are described by nonlinear differential equations, and then a linearized system of stability differential equations with variable coefficients is obtained on the basis of [1, 2]. The variable coefficients reflect the influence of the initial bending state and the variability of the shell thickness. The nonlinear equations of the pre-critical state are solved by the small parameter method for an initial axisymmetric equilibrium mode. An iteration process to determine the critical forces is constructed

by using the small parameter method on a linearized system of stability equations. The problem is solved in three approximations in the small parameters.

1. The nonlinear equations of the pre-critical state of cylindrical shells of variable thickness are [3]

$$M^-(D, w, \Phi) \equiv \Delta(D\Delta w) - (1-\nu) L(D, w) - \frac{1}{R}\frac{\partial^2 \Phi}{\partial x^2} - L(\Phi, w) + N_x \frac{\partial^2 w}{\partial x^2} = 0 \tag{1.1}$$

$$M^+(H, w, \Phi) \equiv \Delta(H\Delta\Phi) - (1+\nu) L(H, \Phi) + \frac{1}{R}\frac{\partial^2 w}{\partial x^2} + \frac{1}{2} L(w, w) = 0$$

$$D = \frac{Eh^3(x, y)}{12(1-\nu^2)}, \qquad H = \frac{1}{Eh(x, y)}$$

$$L(u, v) = \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial x^2} - 2\frac{\partial^2 u}{\partial x\,\partial y}\frac{\partial^2 v}{\partial x\,\partial y}$$

We represent the stress and normal displacement functions as

$$\Phi = \varphi_0 + \varphi(x, y), \quad w = w_0 + w(x, y) \tag{1.2}$$

Here φ_0 and w_0 are the stress and deflection functions corresponding to the pre-critical state of the shell, and $\varphi(x, y)$ and $w(x, y)$ are the increments to these quantities which originate during buckling.

Substituting (1.2) into (1.1) and neglecting second order quantities, we obtain the linearized system of equations

$$\Delta(D\Delta w) - (1-\nu) L(D, w) - \frac{1}{R}\frac{\partial^2 \varphi}{\partial x^2} - L(\varphi, w_0) - L(\varphi_0 w) + N_x \frac{\partial^2 w}{\partial x^2} = -M^-(D, w_0, \varphi_0) \tag{1.3}$$

$$\Delta(H\Delta\varphi) - (1-\nu) L(H, \varphi) + \frac{1}{R}\frac{\partial^2 w}{\partial x^2} + L(w_0, w) = -M^+(H, w_0, \varphi_0)$$

The system (1.3) affords the possibility of finding the solution of the nonlinear system as well as of solving the problem of the stability of the initial bending state. The right-hand sides of (1.3) agree in form with the left-hand sides of (1.1). Hence, when the solution of (1.1) will have been found, the system (1.3) will become homogeneous and will have a nontrivial solution for a fixed value of the load parameter.

Let us assume that the solution of the system (1.1) has been found, then we obtain the stability equations of variable-thickness cylindrical shells in the bending state

$$\Delta(D\Delta w) - (1-\nu) L(D, w) - \frac{1}{R}\frac{\partial^2 \varphi}{\partial x^2} - L(\varphi_0, w) - L(\varphi, w_0) + N_x \frac{\partial^2 w}{\partial x^2} = 0 \tag{1.4}$$

$$\Delta(H\Delta\varphi) - (1-\nu) L(H, \varphi) + \frac{1}{R}\frac{\partial^2 w}{\partial x^2} + L(w_0, w) = 0$$

The equations (1.4) have variable coefficients reflecting the influence of the initial bending state and the variability of the shell thickness. Therefore, the solution of the stability problem of the bending state of variable-thickness cylindrical shells reduces to integrating the system (1.1) and the stability equations (1.4).

Let us assume that the shell thickness can be represented as

$$h = h_0[1 + \varepsilon f(x, y)], \qquad \varepsilon = \frac{h_{\max} - h_{\min}}{2h_0} \tag{1.5}$$

where h_0 is the mean value of the thickness and ε is a small parameter. Then the variable stiffnesses D and H can be written as follows

$$D = D_0[1 + \varepsilon f(x, y)]^3, \quad H = H_0[1 - \varepsilon f(x, y) + \varepsilon^2 f^2(x, y) - \ldots]$$

Substituting these variable quantities into (1.1) and then taking the solution as series expansions in the small parameter ε, we obtain the following solution of the axisymmetric initial equilibrium mode if the shell is hinge-supported at the endfaces $x = 0$ and $x = L$

$$w_0 = f_m \left(\sin \lambda_\mu x + \varepsilon \sum_\rho \alpha_{1\mu\rho} \sin \lambda_\rho x\right) \tag{1.6}$$

$$\varphi_0 = f_m \left(\frac{1}{RH_0\lambda_\mu^2} \sin \lambda_\mu x + \varepsilon \sum_\rho \beta_{1\mu\rho} \sin \lambda_\rho x\right) \tag{1.7}$$

$$\alpha_{1\mu\rho} = \frac{1}{\lambda_\rho^2 (N_{0\rho} - N_{0\mu})} \left(C_{1\mu\rho}^{(1)} - \frac{1}{RH_0D_0\lambda_\rho^2} C_{1\mu\rho}^{(2)}\right)$$

$$\beta_{1\mu\rho} = \frac{\alpha_{1\mu\rho}}{RH_0\lambda_\rho^2} + \frac{1}{\lambda_\rho^4} C_{1\mu\rho}^{(2)}, \qquad N_{0\mu} = D_0\lambda_\mu^2 + \frac{1}{R^2H_0\lambda_\mu^2}$$

Here f_m is the initial pre-critical deflection, $N_{0\rho}$ is a quantity obtained from $N_{0\mu}$ if μ is replaced by ρ, and $C_{1\mu\rho}^{(1)}$ and $C_{1\mu\rho}^{(2)}$ are the right-hand sides of the first approximation equations with the form mentioned in [3]. Depending on the law of thickness variation, we can find $C_{1\mu\rho}^{(1)}$ and $C_{1\mu\rho}^{(2)}$.

Taking account of (1.5) and (1.6), the system (1.4) becomes the following:

$$\begin{gathered}\Delta\Delta w + 3\varepsilon L_\nu^-(f, w) + 3\varepsilon^2 L_\nu^-(f^2, w) + \varepsilon^3 L_\nu^-(f^3, w) + \\ \frac{f_0h_0}{D_0}\left(\lambda_\mu^2 \sin \lambda_\mu x + \varepsilon \sum_\rho \alpha_{1\mu\rho}\lambda_\rho^2 \sin \lambda_\rho x\right)\frac{\partial^2\varphi}{\partial y^2} + \\ \frac{f_0h_0}{D_0}\left(\frac{1}{RH_0}\sin \lambda_\mu x + \varepsilon \sum_\rho \beta_{1\mu\rho}\lambda_\rho^2 \sin \lambda_\rho x\right)\frac{\partial^2 w}{\partial y^2} - \\ \frac{1}{RD_0}\frac{\partial^2\varphi}{\partial x^2} + \frac{N_x}{D_0}\frac{\partial^2 w}{\partial x^2} = 0 \\ \Delta\Delta\varphi - \varepsilon L_\nu^+(f, \varphi) + \varepsilon^2 L_\nu^+(f^2, \varphi) - \varepsilon^3 L^+(f^3, \varphi) + \frac{1}{RH_0}\frac{\partial^2 w}{\partial x^2} - \\ \frac{f_0h_0}{H_0}\left(\lambda_\mu^2 \sin \lambda_\mu x + \varepsilon \sum_\rho \alpha_{1\mu\rho}\lambda_\rho^2 \sin \lambda_\rho x\right)\frac{\partial^2 w}{\partial y^2} = 0 \\ L_\nu^\pm(u^k, v) = \Delta(u\Delta v) - (1 \pm \nu)\, L(u^k, v), \qquad k = 1, 2, 3\end{gathered} \tag{1.8}$$

The system (1.8) contains two small parameters, ε and $f_0 = fm/h_0$. We seek the solution of (1.8) as power series in the small parameters [4]

$$\varphi = \sum_{k=0}^{\infty}\sum_{s=0}^{\infty} \varepsilon^k f_0^s \varphi_{ks}(x, y), \qquad w = \sum_{k=0}^{\infty}\sum_{s=0}^{\infty} \varepsilon^k f_0^s w_{ks}(x, y) \tag{1.9}$$

$$N_x = \sum_{k=0}^{\infty}\sum_{s=0}^{\infty} \varepsilon^k f_0^s N_{ks}$$

Substituting (1.9) into (1.8) and equating coefficients of identical powers of the small parameters, we obtain a system of successive differential equations with constant coefficients

$$M_1(w_{00}, \varphi_{00}) = 0, \quad M_2(w_{00}, \varphi_{00}) = 0 \tag{1.10}$$

$$M_1(w_{10}, \varphi_{10}) = -3L_\nu^-(f, w_{00}) - \frac{N_{10}}{D_0}\frac{\partial^2 w_{00}}{\partial x^2}, \quad M_2(w_{10}, \varphi_{10}) = L_\nu^+(f, \varphi_{00})$$

$$M_1(w_{20},\varphi_{20}) = -3L_\nu^-(f, w_{10}) - 3L_\nu^-(f^2, w_{00}) - \frac{N_{10}}{D_0}\frac{\partial^2 w_{10}}{\partial x^2} - \frac{N_{20}}{D_0}\frac{\partial^2 w_{00}}{\partial x^2}$$
$$M_2(w_{20},\varphi_{20}) = L_\nu^+(f,\varphi_{10}) - L_\nu^+(f^2,\varphi_{00})$$
$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$
$$M_1(w_{01},\varphi_{01}) = \frac{h_0}{D_0}\sin\lambda_\mu x\left(\lambda_\mu^2\frac{\partial^2\varphi_{00}}{\partial y^2} + \frac{1}{RH_0}\frac{\partial^2 w_{00}}{\partial y^2}\right) - \frac{N_{01}}{D_0}\frac{\partial^2 w_{00}}{\partial x^2} \qquad (1.11)$$
$$M_2(w_{01},\varphi_{01}) = \frac{\lambda_\mu^2 h_0}{H_0}\sin\lambda_\mu x\frac{\partial^2 w_{00}}{\partial y^2}$$
$$M_1(w_{02},\varphi_{02}) = -\frac{h_0}{D_0}\sin\lambda_\mu x\left(\lambda_\mu^2\frac{\partial^2\varphi_{01}}{\partial y^2} + \frac{1}{RH_0}\frac{\partial^2 w_{01}}{\partial y^2}\right) - \frac{N_{01}}{D_0}\frac{\partial^2 w_{01}}{\partial x^2} - \frac{N_{02}}{D_0}\frac{\partial^2 w_{00}}{\partial x^2}$$
$$M_2(w_{02},\varphi_{02}) = \frac{\lambda_\mu^2 h_0}{H_0}\sin\lambda_\mu x\frac{\partial^2 w_{01}}{\partial x^2}$$
$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$
$$M_1(w_{11},\varphi_{11}) = -\frac{h_0}{D_0}\sin\lambda_\mu x\left(\lambda_\mu^2\frac{\partial^2\varphi_{10}}{\partial y^2} + \frac{1}{RH_0}\frac{\partial^2 w_{10}}{\partial y^2}\right) - \frac{h_0}{D_0}\sum_\rho\lambda_\rho^2\sin\lambda_\rho x\left(\alpha_{1\mu\rho}\frac{\partial^2\varphi_{00}}{\partial y^2} + \beta_{1\mu\rho}\frac{\partial^2 w_{00}}{\partial y^2}\right) - 3L_\nu^-(f, w_{01}) - \frac{N_{11}}{D_0}\frac{\partial^2 w_{00}}{\partial x^2} \qquad (1.12)$$
$$M_2(w_{11},\varphi_{11}) = L_\nu^+(f,\varphi_{01}) + \frac{\lambda_\mu^2 h_0}{H_0}\sin\lambda_\mu x\frac{\partial^2 w_{10}}{\partial y^2} + \frac{h_0}{H_0}\sum_\rho\alpha_{1\mu\rho}\lambda_\rho^2\sin\lambda_\rho x\frac{\partial^2 w_{00}}{\partial y^2}$$
$$M_1(w_{21},\varphi_{21}) = -3L_\nu^-(f, w_{11}) - 3L_\nu^-(f^2, w_{01}) - \frac{h_0}{D_0}\sin\lambda_\mu x\left(\lambda_\mu^2\frac{\partial^2\varphi_{20}}{\partial y^2} + \frac{1}{RH_0}\frac{\partial^2 w_{20}}{\partial y^2}\right) - \frac{h_0}{D_0}\sum_\rho\lambda_\rho^2\sin\lambda_\rho x\left(\alpha_{1\mu\rho}\frac{\partial^2\varphi_{10}}{\partial y^2} + \frac{\beta_{1\mu\rho}}{RH_0}\frac{\partial^2 w_{10}}{\partial y^2}\right) - \frac{N_{21}}{D_0}\frac{\partial^2 w_{00}}{\partial x^2} - \frac{N_{11}}{D_0}\frac{\partial^2 w_{10}}{\partial x^2} - \frac{N_{20}}{D_0}\frac{\partial^2 w_{01}}{\partial x^2} - \frac{N_{01}}{D_0}\frac{\partial^2 w_{20}}{\partial x^2}$$
$$M_2(w_{21},\varphi_{21}) = L_\nu^+(f,\varphi_{11}) - L_\nu^+(f^2,\varphi_{01}) + \frac{\lambda_\mu^2 h_0}{H_0}\sin\lambda_\mu x\frac{\partial^2 w_{20}}{\partial y^2} + \frac{h_0}{H_0}\sum_\rho\alpha_{1\mu\rho}\lambda_\rho^2\sin\lambda_\rho x\frac{\partial^2 w_{10}}{\partial y^2}$$
$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

Here

$$M_1(u,v) = \Delta\Delta u - \frac{1}{RD_0}\frac{\partial^2 v}{\partial x^2} + \frac{N_{00}}{D_0}\frac{\partial^2 u}{\partial x^2}$$
$$M_2(u,v) = \Delta\Delta v + \frac{1}{RH_0}\frac{\partial^2 u}{\partial x^2}$$

The systems (1.10) — (1.12) agree completely, in structure, with the equations of the theory of circular, constant-thickness cylindrical shells with an initial membrane state. The solution of these equations can be obtained by known methods of cylindrical shell theory.

Therefore, by using the small parameter method to determine the compressive force, an iteration process can be constructed. The solution of the first two equations of the system (1.10) is known in a zero approximation [1].

The corrections to the zero-approximation solution which take account of the variability in the thickness and of the initial bending state will be obtained by solving the system of equations (1.10) — (1.12) successively. The first group of equations (1.10) takes account of the influence of just the variability in thickness, the second group of

Eqs. (1.11) takes account of the initial bending state, and the third group takes account of the mutual influence of the variability in thickness and the bending state.

2. We examine a scheme to determine the critical forces for variable-thickness cylindrical shells when the shell edges are hinge-supported at $x = 0$ and $x = L$.

We take the stress and deflection functions satisfying the boundary condition in the zero approximation as

$$w_{00} = f_{mn} w_{0mn}, \qquad \varphi_{00} = \frac{\lambda_m{}^2 f_{mn}}{RH_0\Delta^2_{mn}} w_{0mn} \tag{2.1}$$

$$w_{0mn} = \sin\lambda_m x \sin\frac{ny}{R}, \qquad \lambda_m = \frac{m\pi}{L}$$

Substituting (2.1) into the first pairs of equations of the system (1.10), we obtain the known value of the compressive force for the initial membrane state [1]

$$N_{00} = D_0\frac{\Delta^2_{mn}}{\lambda^2_m} + \frac{\lambda^2_m}{RH_0\Delta^2_{mn}}, \qquad \Delta^2_{mn} = \left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{R}\right)^2\right]^2 \tag{2.2}$$

This zero-approximation value does not differ at all from the "upper" value of the critical force for circular cylindrical shells of constant thickness h.

Solving the remaining pairs of differential equations of the system (1.10) successively, we find the correction terms to the values in (2.2). We find these correction terms as follows.

The solution of the ks-th approximation satisfying the boundary conditions is taken as

$$w_{ks}(x, y) = \sum_p\sum_q B^{(ks)}_{pq} w_{0pq}, \qquad \varphi_{ks}(x, y) = \sum_p\sum_q A^{(ks)}_{pq} w_{0pq} \tag{2.3}$$

Substituting (2.3) into (1.10) — (1.12) of the ks-th approximation, and then multiplying both sides of the equation by w_{0pq} and integrating over the shell domain, we obtain a system in $B^{(ks)}_{pq}$, $A^{(ks)}_{pq}$. Solving this system, we find

$$B^{(ks)}_{pq} = \frac{1}{\lambda_p{}^2(N_{0pq} - N_{0mn})}\left(C^{(1)}_{pqks} - \frac{\lambda_p{}^2}{RH_0D_0\Delta^2_{pq}} C^{(2)}_{pqks}\right) \tag{2.4}$$

$$A^{(ks)}_{pq} = \frac{\lambda_p{}^2}{RH_0\Delta^2_{pq}} B^{(ks)}_{pq} + \frac{1}{\Delta^2_{pq}} C^{(2)}_{pqks}$$

Setting $m = p$ and $n = q$ in (2.4), we obtain the following conditions to determine the corrections to the values (2.2):

$$C^{(1)}_{mnks} - \frac{\lambda_m{}^2}{RD_0\Delta^2_{mn}} C^{(2)}_{mnks} = 0 \tag{2.5}$$

$$\left(C^{(i)}_{mnks} = \frac{4}{LT}\iint_G F^{(i)}_{ks}(x, y)\, w_{0mn}\, dxdy, \qquad i = 1, 2\right)$$

Here $F^{(1)}_{ks}(x, y)$ and $F^{(2)}_{ks}(x, y)$ are the right-hand sides of the ks-th approximation of the systems (1.10) — (1.12). We find the values of N_{ks} from the condition (2.5) in each approximation.

Therefore, by giving the law of thickness variation, we determine the corrections to the values (2.2) which take account of the bending state and the variability of the shell thickness.

3. As an illustration, let us consider a closed circular cylindrical shell with linearly-varying thickness in the x-axis direction: $h(x) = h_{\min}(1 + \lambda x/L)$. Transforming $h(x)$ in terms of the mean value of the thickness in the form (1.5), we have

$$f(x) = 2x/L - 1, \quad \varepsilon = \lambda/(2+\lambda)$$

$$\alpha_{1\mu\rho} = \frac{48\mu\rho D_0}{\pi^2\lambda_\rho^{\,2}(\mu^2-\rho^2)^2(N_{0\rho}-N_{0\mu})}\left(\lambda_\mu^{\,4} + \frac{\mu^2}{3R^2H_0D_0\rho^2}\right)$$

$$\beta_{1\mu\rho} = \frac{\alpha_{1\mu\rho}}{RH_0\lambda_\rho^{\,2}} - \frac{16\mu\rho}{\pi^2(\mu^2-\rho^2)^2}\frac{\mu^4}{\rho^4}\left(1+\frac{\rho^2-\mu^2}{\mu^2}\right)\frac{1}{RH_0\lambda_\mu^{\,2}}$$

In these expressions $\mu \neq \rho$ and μ as well as $\mu+\rho$ are odd numbers.

Solving the system of equations (1.10), we find $N_{11} = 0$ in a first approximation from condition (2.5), and the coefficients of the series (2.3) are the following:

$$B_{pn}^{(10)} = \alpha_{1mp}^{(10)} f_{mn} \quad (m + p - \text{odd})$$

$$\alpha_{1mp}^{(10)} = \frac{48mp\Delta_{mn}^2 D_0}{\pi^2(m^2-p^2)^2\lambda_p^{\,2}(N_{0pn}-N_{0mn})}\left[1+\frac{\pi^2(p^2-m^2)}{L^2\Delta_{mn}}\right]\times$$
$$\left(1+\frac{\lambda_m^{\,2}\lambda_p^{\,2}}{3R^2H_0D_0\Delta_{pn}^2\Delta_{mn}^2}\right)$$

In the second approximation we have

$$N_{20} = \frac{D_0\Delta_{mn}^2}{\lambda_m^{\,2}}\left(1-\frac{6}{m^2\pi^2}\right)\left(1-\frac{\lambda_m^{\,4}}{3R^2H_0D_0\Delta_{mn}^4}\right)+ \tag{3.1}$$
$$\frac{24(1-\nu)D_0}{L^2\lambda_m^{\,2}}\left(\frac{n}{R}\right)^2\left[1-\frac{1+\nu}{3(1-\nu)}\frac{\lambda_m^{\,4}}{R^2H_0D_0\Delta_{mn}^4}\right]+\eta_{1mpn}$$

$$\eta_{1mpn} = \frac{256}{\pi^2R^2H_0\Delta_{mn}^2L^2}\sum_p\frac{m^4p^2}{(m^2-p^2)^4}\left[1+\frac{\pi^2(p^2-m^2)}{L^2\Delta_{mn}}\right]\left[1-\frac{\pi^2(p^2-m^2)}{L^2\Delta_{pn}}\right]-$$
$$\frac{48D_0L^2}{\pi^4}\sum_p\frac{p\Delta_{pn}^2}{m(m^2-p^2)^2}\left[1+\frac{\pi^2(m^2-p^2)}{L^2\Delta_{pn}}\right]\times$$
$$\left[1+\frac{\lambda_m^{\,2}\lambda_p^{\,2}}{3R^2H_0D_0\Delta_{pn}^2\Delta_{mn}^2}\right]\alpha_{mnp}^{(10)}$$

Let us examine the solution of the second group of equations (1.11). In the first approximation

$$\varphi_{01} = \sum_i\sum_n\frac{\lambda_i^{\,2}}{RH_0\Delta_{in}^2}B_{in}^{(01)}\sin\lambda_i x\sin\frac{ny}{R} - \tag{3.2}$$
$$\frac{1}{2}\left(\frac{n}{R}\right)^2\frac{\lambda_\mu^{\,2}h_0}{H_0}f_{mn}\sin\frac{ny}{R}\left[\frac{\cos(\lambda_\mu-\lambda_m)x}{\Delta_{m-\mu,\,n}^2}-\frac{\cos(\lambda_\mu+\lambda_m)x}{\Delta_{m+\mu,\,n}^2}\right]$$

$$N_{01} = -\frac{8h_0m^2}{\pi RH_0\mu(4m^2-\mu^2)}\left(\frac{n}{R}\right)^2\left(\frac{\lambda_\mu^{\,2}}{\Delta_{mn}^2}+\frac{1}{\lambda_m^{\,2}}\right)-$$
$$\frac{2h_0m\lambda_\mu^{\,2}}{\pi RH_0\mu\lambda_m^{\,2}(4m^2-\mu^2)}\left(\frac{n}{R}\right)^2\left[\frac{(\lambda_\mu-\lambda_m)^2(2m+\mu)}{\Delta_{m-\mu,\,n}^2}+\frac{(\lambda_\mu+\lambda_m)^2(2m-\mu)}{\Delta_{m+\mu,\,n}^2}\right]$$

$$B_{in}^{(01)} = -\alpha_{mni}^{(01)} f_{mn}$$

$$\alpha_{mni}^{(01)} = \frac{2h_0 i\lambda_\mu^2}{\pi R H_0 \lambda_i^2 (N_{0in} - N_{0mn})} \left(\frac{n}{R}\right)^2 \left[\frac{(\lambda_m + \lambda_\mu)^2}{\Delta_{m+\mu,n}^2 [i^2 - (m+\mu)^2]} - \frac{(\lambda_m - \lambda_\mu)^2}{\Delta_{m-\mu,n}^2 [i^2 - (m-\mu)^2]} + \frac{8m\mu\lambda_m^2}{\Delta_{mn}^2 [\mu^2 - (m-i)^2][\mu^2 - (m+i)^2]} + \frac{8m\mu}{\lambda_\mu^2 [\mu^2 - (m-i)^2][\mu^2 - (m+i)^2]} \right]$$

In the second approximation we have

$$\varphi_{02} = \sum_i \sum_n \frac{\lambda_i^2}{RH_0 \Delta_{in}^2} B_{in}^{(02)} \sin\lambda_i x \sin\frac{ny}{R} - \frac{\lambda_\mu^2 h_0}{2H_0} \left(\frac{n}{R}\right)^2 \sum_i^{\infty} B_{in}^{(01)} \sin\frac{ny}{R} \left[\frac{\cos(\lambda_\mu - \lambda_m)x}{\Delta_{\mu-i,n}^2} - \frac{\cos(\lambda_\mu + \lambda_m)x}{\Delta_{\mu+i,n}^2} \right] \tag{3.3}$$

$$N_{02} = \frac{2mh_0^2\lambda_\mu^4}{\pi R H_0 \lambda_m^2 \mu (4m^2 - \mu^2)} \left(\frac{n}{R}\right)^4 \left(\frac{2m+\mu}{\Delta_{m-\mu,n}^2} + \frac{2m-\mu}{\Delta_{m+\mu,n}^2} \right) - \eta_{2mni\mu}$$

$$\eta_{2mni\mu} = \frac{2\lambda_\mu^2 h_0 m}{\pi R H_0 \lambda_m^2} \left(\frac{n}{R}\right)^2 \sum_i \left[\frac{8i\lambda_i^2\mu}{\Delta_{in}^2 [\mu^2 - (m-i)^2][\mu^2 - (m+i)^2]} + \frac{8\mu i}{\lambda_\mu^2 [\mu^2 - (m-i)^2][\mu^2 - (m+i)^2]} + \frac{(\lambda_m + \lambda_\mu)^2}{\Delta_{m+i,n}^2 [m^2 - (\mu+i)^2]} - \frac{(\lambda_m - \lambda_\mu)^2}{\Delta_{m-n,i}^2 [m^2 - (\mu-i)^2]} \right] \alpha_{mni}^{(01)}$$

Here $m + i$ are even numbers.

Solving the third group of equations (1.12), we find the mutual effect of variability of the thickness and the initial bending state on the magnitude of the critical force.

In the first approximation we find $N_{11} = 0$, $B_{jn}^{(11)} = \alpha_{mnj}^{(11)} f_{mn}$.

The values of N_{12} and N_{21} are not presented because of the awkwardness of the expressions. Therefore, the series (1.9) becomes in three approximations

$$N_x = N_{00} + \varepsilon N_{10} + \varepsilon^2 N_{20} + f_0 N_{01} + f_0^2 N_{02} + \varepsilon f_0 N_{11} + \varepsilon^2 f_0 N_{21} + \varepsilon f_0^2 N_{12} \tag{3.4}$$

Varying m and n we find the last value of N_x. The remaining parameters are determined so that $m + p$ would be odd numbers and $m + i$ even numbers. The greatest influence of the pre-critical bending state occurs for a value of μ close to the corresponding axisymmetrical buckling mode, i.e.

$$\mu = \frac{L}{\pi R} \sqrt{\frac{R}{h_0}} \sqrt[4]{12(1-\nu^2)}$$

Moreover, the influence of the bending state increases as m approaches $\mu/2$. Since μ is odd, then $\mu = 2m - 1$. Therefore, in seeking the least value of N_x it is sufficient to vary n. The number of waves along the arc can be taken in the order of $\sqrt{R/h_0}$.

As an illustration, let us consider a shell with the following geometric and physical parameters

$L/R = 2$, $R/h_0 = 180$, $\nu = 0.3$, $h_{max} = 2h_{min}$, $= 1/3$, $h_0 = 1{,}5\ h_{min}$

Presented in Table 1 are the dimensionless values of the critical forces of variable-thickness cylindrical shells ($h_{max} = 2h_{min}$) as a function of the initial bending state ($\mu = 23$, $m = 12$, $n = 14$) for the zero, first and second approximations $N_x^{(0)} = N_{00}^* + \varepsilon N_{10}^* + \varepsilon^2 N_{20}^*$, $N_x^{(1)} = N_x^{(0)} + f_0 N_1$, $N_x^{(2)} = N_x^{(1)} + f_0^2 N_2$, $(N_1 = N_{01}^* + \varepsilon^2 N_{21}^*$, $N_2 = N_{02}^* = \varepsilon N_{12}^*)$; $N_x^* = N_x R / E h_{min}^2$.

Table 1

f_0	Zero approximation	$f_0 N_1$	$f_0^2 N_2$	First approximation	Second approximation
0.2	1.552	—0.389	0.029	1.163	1.192
0.3	1.552	—0.587	0.072	0.965	1.037
0.4	1.552	—0.776	0.119	0.766	0.885
0.5	1.552	—0.972	0.182	0.580	0.762
0.6	1.552	—1.172	0.265	0.380	0.645

Table 2

$\frac{h_{max}}{h_{min}}$	Zero approximation	εN_{10}^*	$\varepsilon^2 N_{20}^*$	Second approximation
1.22	0.738	—	0.006	0.744
1.50	0.944	—	0.028	0.975
1.86	1.234	—	0.084	1.318
2.33	1.733	—	0.198	1.931
3.0	2.420	—	0.448	2.868

Table 3

f_0	Zero approximation	$f_0 N_{01}^*$	$f_0^2 N_{02}^*$	First approximation	Second approximation
0.2	0.640	—0.171	0.012	0.469	0.481
0.3	0.640	—0.257	0.027	0.383	0.410
0.4	0.640	—0.342	0.047	0.298	0.345
0.5	0.640	—0.427	0.075	0.213	0.288
0.6	0.640	—0.513	0.110	0.127	0.237

Let us examine some particular cases of the problem.

Cylindrical shell of variable thickness for an initial membrane state. In this case, setting $f_0 = 0$ into (3.4), we obtain the values of the upper critical forces which are presented in Table 2 as a function of the thickness ratio h_{max}/h_{min} ($m = 12$, $n = 10$).

Constant thickness cylindrical shell for an initial bending state. In this case, setting $\varepsilon = 0$ into (3.4), we obtain the values of the critical forces as a function of the initial bending state, presented in Table 3 ($\mu = 23$, $m = 12$, $n = 14$). The shell thickness is taken equal to the minimum value h_{min} of a variable thickness shell

($h = h_{min}$).

As the numerical examples show, the difference between the first and second approximations is negligible if f_0 and $\varepsilon \leqslant 0.6$. Hence, the three approximations in the form of (3.4) in the small parameter reach a satisfactory approximation in the solution of stability problems of the initial bending state.

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